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Modeling Portfolio Return and Risk using GARCH

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Abstract

This study explores the application of GARCH and GJR-GARCH models in measuring portfolio return and risk using high-frequency daily return data from the S&P 500 and DAX indices over the period 2000–2020. The research aims to construct a parsimonious and practical volatility forecasting framework by integrating both symmetric and asymmetric shock models, as well as parametric and non-parametric approaches. The methodology follows foundational works by Engle (1982), Bollerslev (1986), and Glosten et al. (1993), and incorporates Value-at-Risk (VaR) to assess potential portfolio losses. Empirical results reveal that actual return distributions exhibit fat tails and skewness, invalidating the normality assumption and necessitating the use of Student's t-distributions for better accuracy. The GJR-GARCH model proves superior in capturing asymmetric shock effects, as evidenced by higher predictive performance and alignment with observed volatility dynamics, particularly during financial turmoil such as the 2008 crisis. Furthermore, ARMA-enhanced GJR-GARCH models yield better fitting results, especially in forecasting conditional mean and variance. The findings indicate that DAX required higher risk compensation in multiple periods compared to S&P 500, which showed concentrated volatility during the crisis period only. The study highlights the practical implication of selecting appropriate volatility models for dynamic risk management and suggests the inclusion of autoregressive components to enhance return prediction accuracy. Ultimately, this research contributes to the advancement of financial econometric modeling by demonstrating the efficacy of advanced GARCH frameworks in real-world portfolio risk estimation.

Keywords: Portfolio Return, GARCH Model, GJR-GARCH, Volatility Forecasting, Value-at-Risk (VaR), Financial Risk Management

Introduction

Portfolio management is a dynamic field in financial risk management with its diverse behaviour. Its historical development goes back to early 1950s (Markowitz, 1952). Financial asset return volatilities and correlations are key components of measuring the risks of a portfolio. According to Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X. (2013) volatility has persistent dynamics; it changes over time and even across financial assets and asset classes.

Here, we employ a practical method suggested by Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X. (2013) and incorporate previous methodologies developed by Bollerslev (1986) and GJR-GARCH Glosten et al. (1993) to measure associated risks of S&P 500 and DAX. Our primary objective is to arouse affinity between practice and theory by drawing a better parsimonious model.

We stress the GARCH models with different scenarios in order to highlight model features. Our first concern is modelling level; we use portfolio level data (aggregated). Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X. (2013) covers the topic about the distinction of aggregated and asset level data and clearly indicate that in general risk measurement requires a portfolio data.

Secondly, we consider about the frequency of the observations, we employ high frequency data in order to achieve a real volatility prediction. We also deal with parametric and non-parametric volatility estimation, but mostly we emphasize non-parametric methods with different distribution assumptions.

Our third concern is to examine the transmission of dynamic return and volatility by using one of univariate GARCH models. Various practitioners have investigated the linkage of return and volatility, our model is optimized based on Bollerslev (1986), Bollerslev, Engle and Wooldridge (1988) and GJR-GARCH Glosten et al. (1993). Moreover, we also consider simultaneous shock transmission of return series and volatilities by following Arouri, Lahiani and Nguyen (2015) and GJR-GARCH Glosten et al. (1993).

Finally we follow Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X (2013) GARCH model incorporate with value at risk (VaR) quantile risk measurer in order to estimate the level of portfolio loss for our empirical analysis.

We practically focus on building a parsimonious model for a portfolio level data. Information dependencies are key features when optimizing portfolio strategy, it is essential because of financial asset's time varying characteristics. Thus, we consider symmetric and asymmetric shocks to volatility estimation and different distribution assumptions of return series as employed in Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X (2013). We explore more advanced concepts during the estimation of portfolio return loss such as complex constraints, methods for estimating moments and visualisation for better understanding the optimisation problems. Thus, to solve a portfolio optimization in the mean and variance framework.

Data overview

Our sample data frame covers the periods between 01 January 2000 and 31 December 2020. We collected daily time series of United States (US) and Germany stock market portfolio indices, namely S&P 500 and DAX over the mentioned period. Obtained data are downloaded from Yahoo Finance, original source for S&P 500 is NYSE or NASDAQ and for DAX is Frankfurt Stock Exchange databases. Daily price series of the data is collected in order to have a high frequency of data for adequately observing the behaviour and level of volatility during the observation period. Denominated currency for price series are local currencies of the countries, namely USD and EURO. Daily returns are calculated in a simple way, respectively for each index. Table 1 provides statistical and stochastic properties of the employed time series returns data.

Table 1. Summary statistics

Statistics	S&P 500	DAX
Observations	4779	4794
Mean	0.0002	0.0002
Max.	0.1158	0.1140
Min.	-0.0903	-0.0849
Standard deviation	0.0121	0.0148
Skew.	-0.0246	0.0920
Kurt.	11.6834	7.8051
Jarque-Bera	15015***	4646***

Note: *$p < 0.1$; **$p < 0.05$; ***$p < 0.01$.

Table 1 reports basic selected statistics of employed return data for S&P 500 and DAX. As shown above average daily return series are close to zero, the other statistic properties such as maximum and minimum quantify the low and high value of the data over the sample period respectively for S&P 500 and DAX. Standard deviation gives a good indication about average daily risk of returns. Skewness and Kurtosis coefficients are important in order to measure asymmetry and shape of the return series. They will help us to build more holistic view of risk. For instance, skewness can measure how much a return distribution leans to the left or right. Negative skew is a right-leaning curve while a positive skew is a left-leaning curve. Kurtosis is a measure of thickness of the tails of a return distribution. With negatively skewed coefficients (skewness is less than 1) and kurtosis higher than three for both indices, the S&P 500 and DAX returns show asymmetric and fatter tails data. Which means extreme negative and positive returns are more common for these datasets.

Methodology

Econometric set-up and methodology

We start by defining what the GRACH models are. GARCH stands for Generalized Autoregressive Conditional Heteroscedasticity. It is a popular approach to model volatility in financial risk management, portfolio management and investment world.

Before GARCH, another model called Auto Regressive Conditional Heteroscedasticity (ARCH) models existed. The model process introduces conditional heteroscedasticity wherein data shows time dependent varying characteristic and is unpredictable. Moreover, variance, in the model, is not constant and remains conditional on the past with auto regressive behaviour (Engel, 1982).

The GARCH analysis starts with times series of returns, each return is observed at a regular frequency, like daily, weekly and so on. In our analysis, we use daily frequency of return series for both S&P 500 and DAX. Assume at t – 1, we want to predict the next return R_t. For this, we can use the information set consisting of all the past and current returns available as of time t – 1. Thus, to compute the expected returns we follow Engle (1982), Bollerslev (1986) and Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X (2013).

Expected return:

$$\mu\_t = \sum [R\_t \mid I\ (t-1)] \qquad (1)$$

Where:

$I_{\ t-1}$ – is the information set available at time of prediction (t – 1),

μ_t – is mean prediction return.

The prediction is of course not expected to be perfect; there is a prediction error. We follow Engle (1982) to calculate the prediction error as:

$$e_t = R_t - \mu_t, \qquad (2)$$

Similarly, we can predict volatility of time t as expected variance based on the information at t – 1. Thus, we follow Engle (1982) and Bollerslev (1986) to calculate expected volatility.

$$\sigma_t^2 = var\ [R_t \mid I(t-1)] \qquad (3)$$

Volatility is not directly observable, but it is related to prediction error. As discussed earlier, if prediction works well, the residual should equal to volatility multiplied by a random variable from a white noise process. The equation below indicates correlation of volatility to the residuals (Bollerslev, 1986).

$$e_t = \sigma_t * \zeta(\text{white noise}) \qquad (4)$$

To transform the above expectations into practice, we define ARCH equation by following Engle (1982). The equation is designed to show how future variance is more affected by recent events rather than distant ones. By giving more weight to the most recent observation, ARCH equation achieves a higher forecast accuracy. Thus, ARCH components are the predicted variance equals a constant omega plus a weighted sum of the p most recent observed squared prediction errors (Engel, 1982).
As mentioned earlier, GARCH models are invented based on ARCH. GARCH is a famous model which is more commonly used by researchers.

$$GARCH(p,q): \sigma_t^2 = \omega + \sum_{i=1}^{p} a_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \quad (5)$$

$$GARCH(1,1): \sigma_t^2 = \omega + \alpha e_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (6)$$

The GARCH (p,q) model equation

Besides p-period lags of residuals, GARCH models add q-period lags of variance for predicting the current variance. Hence the basic GARCH(1,1) model states, the variance of time t is a sum of three components: a constant omega, alpha times residuals squared of time t – 1, and beta times variance of time t – 1. Figure 1 below illustrates the difference of variance prediction between GARCH and ARCH.

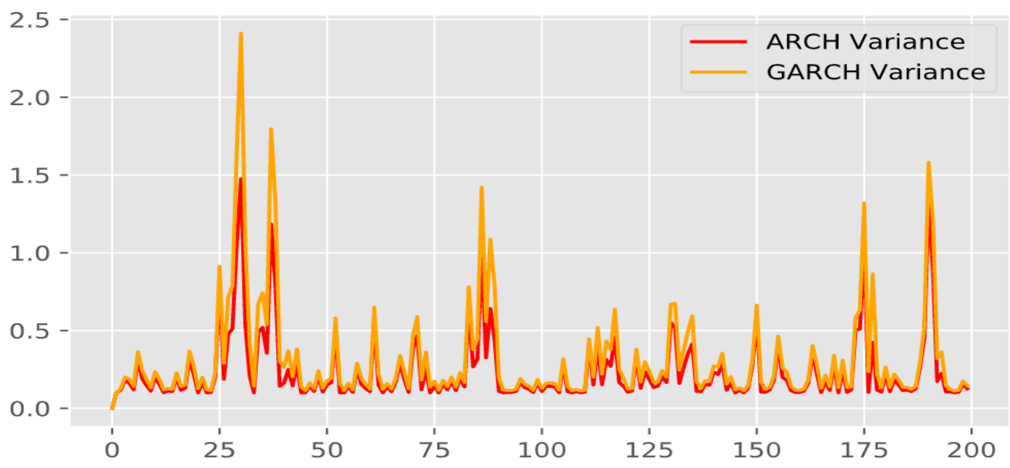

Figure 1. Difference of ARCH and GARCH volatility.

As shown in Figure 1 above, GARCH model generates higher volatility due to additional moving average component of beta multiplied by lag 1 variance. In addition, ARCH shows a short run persistence of the past shock while GARCH effects to the long run persistence of past volatility.

GARCH models can be understood intuitively. First, the model is autoregressive in nature. It estimates volatility at time t on the bases of information known as of t – 1. Second, it estimates volatility as a weighted average of pass information. For making a GARCH (1,1) process realistic, we further adjust two types of parameter restrictions based on Andersen, T. G., Bollerslev, T., Christoffersen, P. F., & Diebold, F. X (2013).

Results and Discussion

In this part, we discuss obtained results of our analysis from GARCH (1,1) model. The model performance and some further parameter adjustments will also be presented. In addition, we compare gained results with another competing GJR-GRACH (1,1) model to have a stronger analysis.
In order to define a good model, we need to have assumptions that fits most representative of the actual data since volatility is not directly observable quantity and is estimated through price return fluctuations. From equation (7) returns equal:

$$R_t = \mu_t + e_t \qquad (7)$$

As shown in equation 7, returns are equal to mean return and residual (prediction error). Where the residuals are stochastic return shocks dependent on the size of the volatility as shown in equation (3). Under the standard GARCH (1,1) model, prediction error is normally distributed with a zero mean and its correlated volatility. However, normal distribution is not realistic when analysis time series stock return data. To account this, GARCH models require making distribution assumption of both residuals and mean return. In equation 9, prediction error is normally distributed and dependent on the size of volatility. This implies that if prediction error in equation 7 is divided by the model-estimated volatility, we obtain standardized returns. In other words, model residuals divided by conditional volatility. Based on Andersen, Bollerslev, Diebold and Labys (2000) we compute standardized returns with underlying formula:

$$Z_t = \frac{R_t - \mu_t}{\sigma_t} \qquad (8)$$

To test this approach, we use histogram of actual returns and standardized returns in order to review their distribution curve. Below Figure 5 illustrates the difference in actual and standardized returns in histograms. By comparing the density function of the normal distribution in histogram of standardized returns 2 and 4 we observe non-normal distribution of returns for both DAX and S&P 500. In grey and black distribution of return series, the blue line is the normal distribution, while in red the actual ones. Compared to the normal the actual return distribution is much more peaked around zero.

In the tails, we see that extreme observations occur and that normal distribution cannot fit them, as it assigns them a zero density. Therefore, actual return distribution has fat tails and that happens frequently in the price return data.

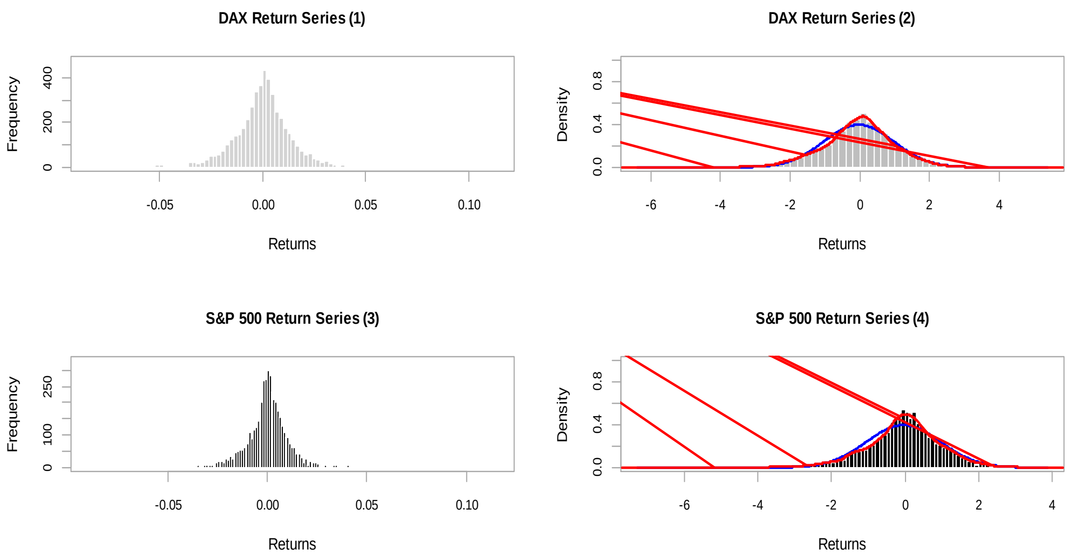

Figure 2. Difference of actual and standardized returns in histogram

The standard GARCH model uses the squared prediction error e_t^2 to forecast the return variance using one single equation. In other words, standard GARCH model assumes symmetric shocks on volatility. Positive and negative changes in price returns would have the same impact on volatility. Therefore, it does not distinguish between positive and negative prediction errors.

However, in reality the size of the prediction error matters. To capture positive and negative changes we need to model asymmetric shocks on volatility. For instance, when things are good, price goes up slowly and steadily; when things turn bad asymmetric shocks arise like during the financial crisis, everyone panics and therefore prices take a sharp plunge. GJR-GARCH is developed to address the asymmetric shock effect on volatility. We follow GJR-GARCH model by Glosten et al. (1993) as specified below to indicate the size of the prediction error:

$$\sigma_t^2 = \begin{cases} \omega + (\alpha+\gamma)e_{t-1}^2 + \beta\sigma_{t-1}^2 & : e_{t-1} \leq 0 \\ \omega + \alpha e_{t-1}^2 + \beta\sigma_{t-1}^2 & : e_{t-1} > 0 \end{cases} \quad (9)$$

With: $\gamma \geq 0$

GJR-GARCH equation is very similar to the standard GARCH model, except it adds a conditional parameter. When shocks are negative to return, the conditional parameter will be included in the equation to account the additional impact. This means that we need to apply a larger multiplier on the squared prediction error. Formally, using as a coefficient $\alpha+\gamma$ instead of α with the additional γ parameter being positive. These two abovementioned equations together define the GJR-GARCH model. We then use this model to analyse the GARCH dynamics for the daily returns of each index.

Table 2. Estimation coefficients of standard GARCH and GJR – GARCH

Parameters	S&P 500		DAX	
	standard GARCH	**GJR-GARCH**	**standard GARCH**	**GJR-GARCH**
μ	0.0005*** (0.0001)	0.0002* (0.0001)	0.0006*** (0.0001)	0.0002* (0.0001)
ω	0.0000* (0.0000)	0.0000* (0.0000)	0.0000 (0.0000)	0.0000* (0.0000)
α	0.1035*** (0.0150)	0.0000 (0.0096)	0.0848*** (0.0160)	0.0000 (0.0082)
β	0.8943*** (0.0137)	0.8913*** (0.0144)	0.9096*** (0.0163)	0.9124*** (0.0139)
γ	---	0.0199*** (0.0263)	---	0.0156*** (0.0227)
Skew.	0.9214*** (0.0173)	0.8881*** (0.0173)	0.9316*** (0.0175)	0.9100*** (0.0179)
Shape	6.8145*** (0.5107)	7.9569*** (0.9686)	9.7260*** (0.0237)	11.5554*** (1.8674)
Observations	4779	4779	4794	4794
Log Likelihood	15579	15675.82	14401.83	14488.24

Note: *p < 0.1; **p < 0.05; ***p < 0.001

Based on Engle and VK. NG (1993) we generated new impact curve plots for each index in order to visualize the impact of news on volatility estimation. As shown in Figure 5, the impact of squared error on the variance prediction after a negative error is 0.0199 and 0.0156 respectively for S&P 500 and DAX, which is the sum of alpha and gamma. Comparing to the value of the positive error (see Appendix B for the calculation), we see that the response of the variance after a negative surprise in returns have much higher asymmetry for each index as shown in the news impact curve plots (Figure 4). Engle and VK. NG (1993) found that GJR-GARCH is the best model to measure the effect of new information on volatility estimation.

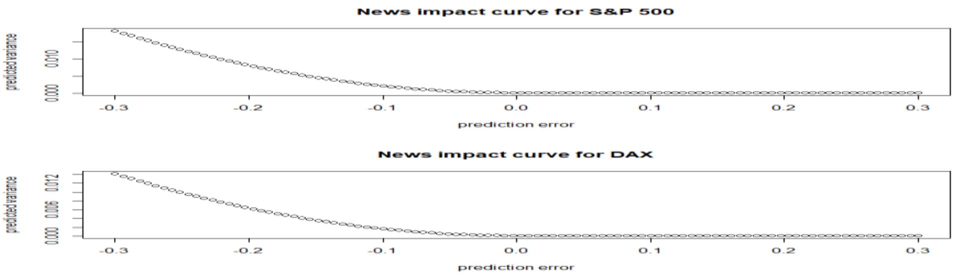

Figure 3. News impact curve for S&P 500 and DAX

Moreover, Figure 4 demonstrates the compression of standard GARCH and GJR-GARCH volatilities respectively for S&P 500 and DAX. In grey, the distribution of absolute value of actual return series, and red and blue lines represent the volatility dynamics, namely red is GJR-GARCH while blue is standard GARCH.

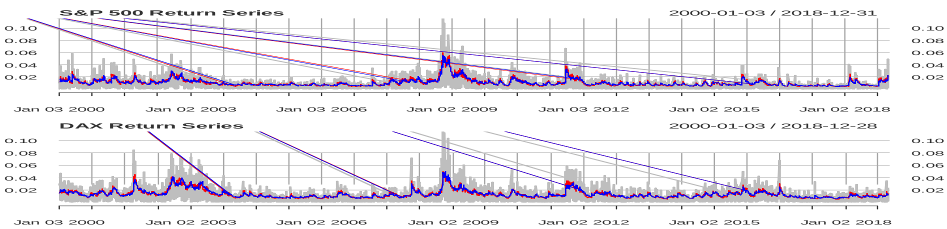

Figure 4. Volatilities of standard GARCH and GJR-GARCH.

As we observe in the above graphs, GJR-GARCH generated volatility with skewed student t distribution is more in line with the movement of actual return series compared to the normal distribution.

Estimated risk and reward coefficients allow us to plot the series of predicted returns over the sample period, in order to visualize the periods which investors require higher returns. Below in Figure 8, we note the spike is around 2008 during the financial crises for both indices. For DAX, investors required higher returns in order to compensate higher risk between the period 2000 and 2003.

Moreover, to estimate the exact level of return and risk we further need to advance the model. Based on Ferenstein and Gasowski (2004) we use the Auto Regression (AR) approach where the model exploits the correlation between today's return and tomorrow's return.

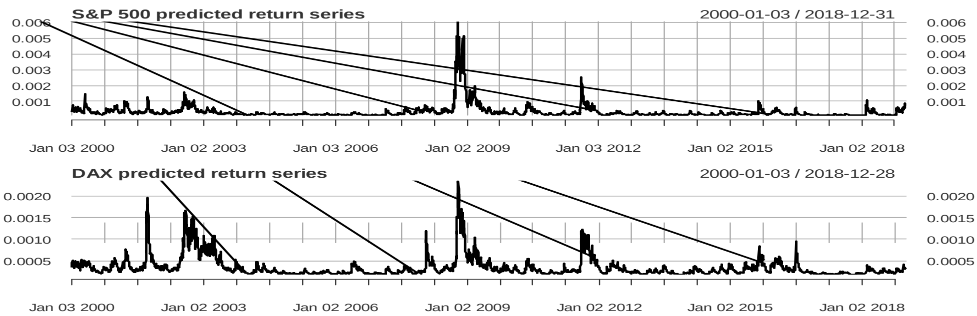

Figure 5. Predicted return series for S&P 500 and DAX.

Table 3. Estimation coefficients of AR(1) GJR-GARCH and ARMA(1,1) GJR - GARCH for S&P 500 and DAX.

Parameters	S&P 500 AR(1) GJR-GARCH	S&P 500 GJR-GARCH ARMA(1,1)	Parameters	DAX AR(1) GJR-GARCH	DAX GJR - GARCH ARMA(1,1)
μ	0.0002** (0.0001)	0.0002** (0.0000)	μ	0.0003* (0.0001)	0.0002** (0.0001)
AR(1)	-0.0657*** (0.0144)	0.5308*** (0.1554)	AR(1)	-0.0181 (0.0146)	0.6965*** (0.1216)
MA(1)	---	-0.5982*** (0.1485)	MA(1)	---	-0.7125*** (0.1189)
ω	0.0000* (0.0000)	0.0000* (0.0000)	ω	0.0000* (0.0000)	0.0000* (0.0000)
α	0.0000 (0.0090)	0.0000 (0.0086)	α	0.0000 (0.0081)	0.0000 (0.0080)
β	0.8952*** (0.0136)	0.9006*** (0.0131)	β	0.9132*** (0.0141)	0.9145*** (0.0144)
γ	0.1893*** (0.0248)	0.1753*** (0.0260)	γ	0.1537*** (0.0230)	0.1501*** (0.0243)
Skewness	0.8804*** (0.0173)	0.8733*** (0.0177)	Skewness	0.9074*** (0.0180)	0.9057*** (0.0181)
Shape	7.9293*** (0.9589)	7.8784*** (0.9491)	Shape	11.5163*** (1.8641)	11.5369*** (1.8751)
Observations	4779	4779	Observations	4794	4794
Log-likelihood	15675.82	15690.57	Log-likelihood	14489.05	14489.44

Note: *p0.1; **p < 0.05; ***p < 0.001.

As more adjustment we make to the GARCH model, our model gets more reliable and makes prediction that fits well with the observed returns. Next, we evaluate the accuracy of employed GARCH model specifically for its mean and variance distribution. For the mean, we use the equation (7) where the prediction error is defined as the difference between actual and predicted return. We achieve a good fit on our analysis when the mean prediction error is small.

Conclusions

We attempted to express the potential capacity of a dynamic GARCH models in the field of practical financial econometrics. We inspected a large amount of literature on volatility modelling and their practical relevance in portfolio risk measurement and management. The implication of our discussion is GARCH models incorporate with VaR offers a comprehensive parsimonious framework for successfully measuring the downside risks of portfolio returns and GARCH models itself for successfully modelling the dynamic features of portfolio returns respectively for S&P 500 and DAX.

By applying standardized returns method, we solve the issue of normal distribution in portfolio returns that enabled us to observe different volatility transmission in different periods. Moreover, we employed a high frequency of return data for both indices and thus enabled us more accurate risk assessment for practical volatility forecast.

Our findings from this analysis suggests that investors for S&P 500 only required higher returns around 2008 during the financial crises, while those for DAX required more compensation not only in 2008 but also from 2000 to 2003 and from 2015 to 2016 in order to cover losses as shown in Figure 8.

Moreover, Table 3 clearly indicates that DAX returns show statistical significant coefficient for AR(1) GJR-GARCH, which means stock prices of DAX does not really depend on its past prices. However, we observe a strong AR(1) for S&P 500 returns implying stock prices of S&P 500 strongly requires the lag 1 auto regression .

Finally, after discussing many implications of portfolio risk management in volatility estimation, we observed a better parameter coefficients with ARMA(1,1) GJR-GARCH. Our estimated volatility forecast with this model presented a good fit to the actual volatility series of both indices as proved in the results of this article.

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